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Orthonormal polynomials on the unit circle and spatially discrete Painlevé II equation

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Abstract. We consider the polynomials $\phi_n(z) = \kappa_n(z^n + b_{n-1}z^{n-1} + \cdots)$ orthonormal with respect to the weight $\exp(\sqrt{\lambda}(z + 1/z))\,\mathrm{d}z/2\pi\mathrm{i}z$ on the unit circle in the complex plane. The leading coefficient κ_n is found to satisfy a difference-differential (spatially discrete) equation which is further proved to approach a third-order differential equation by double scaling. The third-order differential equation is equivalent to the Painlevé II equation. The leading coefficient and second leading coefficient of $\phi_n(z)$ can be expressed asymptotically in terms of the Painlevé II function.

1. Introduction

In this paper, we discuss the orthonormal polynomials with respect to the weight $\exp(\sqrt{\lambda}(z + 1/z))\,\mathrm{d}z/2\pi\mathrm{i}z$ on the unit circle in the complex plane [1],

$$\oint \phi_n(z)\,\overline{\phi_m(z)}\,\mathrm{e}^{\sqrt{\lambda}(z+1/z)}\frac{\mathrm{d}z}{2\pi\mathrm{i}z} = \delta_{nm} \qquad m, n \geqslant 0 \tag{1.1}$$

where the integral is over the unit circle, and $\overline{\phi_m(z)}$ means the complex conjugate of $\phi_m(z)$, λ is a positive parameter. The polynomials $\phi_n(z) = \kappa_n z^n + \cdots$ have an explicit representation in terms of the Toeplitz determinants $D_n(\lambda) = D_n(\exp(2\sqrt{\lambda}\cos\theta))$ [1]. Here we are interested in analysing the properties of the leading coefficient $\kappa_n(\lambda)$. As discussed in [1], κ_n^2 can be expressed in terms of the Toeplitz determinants,

$$\kappa_n^2 = \frac{D_{n-1}(\lambda)}{D_n(\lambda)} \tag{1.2}$$

and $\kappa_n^2(\lambda) \to 1$, as $n \to \infty$ for any λ (see (12.3.19) and (10.2.4) in [1]).

From the orthonormal property and the recursion formula of the ϕ_ns, we show that κ_n satisfies the following difference-differential equation

$$\frac{n+1}{2s}\frac{(\kappa_n^2)_s}{\kappa_n^2} - \frac{\kappa_{n-1}^2 - \kappa_n^2}{\kappa_n^2} + \frac{1}{4}\frac{(\kappa_{n+1}^2)_s}{\kappa_{n+1}^2}\frac{(\kappa_n^2)_s}{\kappa_{n+1}^2 - \kappa_n^2} - \frac{1}{4}\left(\frac{(\kappa_n^2)_s}{\kappa_n^2}\right)^2 = 0 \tag{1.3}$$

where $s = \sqrt{\lambda}$, and $(\kappa_n^2)_s = (\mathrm{d}/\mathrm{d}s)(\kappa_n^2)$. If we make the ansatz

$$\kappa_n^2(s) = 1 + \frac{c_1}{(n+1)^\alpha}R(T(n,s)) \tag{1.4}$$

with $T(n,s) = t(n,s) + \epsilon(n,s)$, and

$$t(n,s) = \frac{(n+1)^\beta}{c_2}\left(\frac{c_3\, s}{n+1} - 1\right) \tag{1.5}$$

then we obtain, as $n, s \to \infty$, and $c_3\, s/(n+1) \to 1$, if we choose $c_3^2 = 4$, the other parameters then satisfy $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $c_1 = -2^{1/3}$, $c_2 = -1/2^{1/3}$, $\epsilon(n,s) = \mathrm{O}(1/(n+1)^{1/3})$, and the function $R(t)$ satisfies

$$(R'')^2 - 8(R')^3 + 4t(R')^2 - 2R'R''' + \mathrm{o}(1) = 0. \tag{1.6}$$

If we drop the o(1) term, this equation is another form of the Painlevé II equation discussed by Tracy and Widom in [2]. Since equation (1.3) contains a continuous independent variable s, and a discrete independent variable n, let us call (1.3) a spatially discrete Painlevé II equation to distinguish from the discrete Painlevé II equation [8] which has one discrete independent variable.

Therefore, we have another proof of the asymptotic formula

$$\kappa_n^2 = 1 - \frac{2^{1/3}}{(n+1)^{1/3}} R(t) + \cdots \tag{1.7}$$

where $R'(t) = -q^2(t)$, and $q(t)$ satisfies the Painlevé II equation $q'' = tq + 2q^3$. This asymptotic formula was first proved by Baik *et al* [3] by studying the corresponding Riemann–Hilbert problems. They discussed much more asymptotic properties for κ_n (see [3] for details). The asymptotic formula (1.7) is used for investigating the distribution of the length l_n of the longest increasing subsequence of a random permutation [3], by discussing the asymptotics of $D_{n-1}(\lambda) = \mathrm{e}^{\lambda} \prod_{k=n}^{\infty} \kappa_k^2$. The distribution of l_n (as $n \to \infty$) is the same as the distribution of the largest eigenvalue of the Gaussian unitary ensemble (GUE) in random matrix theory [3–5]. In [5], Tracy and Widom used a different method to study the distribution of l_n by investigating the asymptotics of $D_n(\lambda)$ directly. They also obtained the distribution of the length of the longest increasing subsequence of an odd permutation [5].

In [6], Hisakado discussed the same polynomials as in this paper, and he also obtained the Painlevé II equation from a discrete equation which is called the discrete string equation in [6]. The difference is as follows. In [6], Hisakado discussed the fact that the constant term of the polynomial $\phi_n(z)/\kappa_n$ satisfies a discrete equation (discrete string equation). In this paper, we discuss κ_n, the coefficient of the leading term of $\phi_n(z)$, which satisfies a spatially discrete equation (difference-differential equation). And the discrete string equation in [6] is convergent to the original Painlevé II ($q'' = xq + 2q^3$). In this paper, the spatially discrete equation is convergent to a third-order differential equation. As discussed by Tracy and Widom in [2], this third-order equation is equivalent to the Painlevé II equation.

It is known that the Painlevé equations have discrete analogues. See [7–10] and references therein for the discrete or q-difference Painlevé equations. In this paper, we show that the Painlevé II equation has a spatially discrete version. So far, to my knowledge, the spatially discrete versions for Painlevé equations are not known very well. In [7], Fokas *et al* discussed another spatially discrete integrable equation, which is obtained from orthonormal polynomials on the real line satisfying a recursion formula in the form (2.11). It is discussed in [7] that the simultaneous solution of the spatially discrete equation in [7] and of the discrete Painlevé I equation solves a special case of the Painlevé IV equation.

This paper is organized as follows. In the next section, we state the recursion formula for the orthonormal polynomials on the unit circle, which is proved in [1]. In section 3, we use the recursion formula and the orthonormal property of the polynomials to investigate the leading coefficient $\kappa_n(s)$, which is found to satisfy a difference-differential equation, or spatially discrete equation. And we also obtain a formula for the second leading coefficient of $\phi_n(z)$ in terms of κ_n and κ_{n-1}. In section 4, the difference-differential equation is proved to tend to the second Painlevé equation in a third order equation form. Then we obtain the asymptotics of the coefficients for the leading term and for the second leading term of $\phi_n(z)$.

2. Recursion formula for the polynomials

Let us consider the orthonormal polynomials $\phi_n (n = 0, 1, 2, \ldots)$ on the unit circle in the complex plane defined by (1.1). If we let $f(\mathrm{e}^{\mathrm{i}\theta}) = \exp(2\sqrt{\lambda}\cos\theta) = \exp(\sqrt{\lambda}(z + 1/z))$, $z = \mathrm{e}^{\mathrm{i}\theta}$, then (1.1) becomes

$$\int_{-\pi}^{\pi} \phi_n(\mathrm{e}^{\mathrm{i}\theta})\,\overline{\phi_m(\mathrm{e}^{\mathrm{i}\theta})}\,f(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi} = \delta_{nm} \qquad m, n \geqslant 0. \tag{2.1}$$

Since $2\sqrt{\lambda}\cos\theta$ is even in θ, the coefficients of ϕ_n are real [1]. It is also proved in [1] that ϕ_n satisfy the recursion formulae

$$\kappa_{n-1} z\phi_{n-1}(z) = \kappa_n\phi_n(z) - \phi_n(0)\,\phi_n^*(z) \tag{2.2}$$

$$\kappa_n\phi_{n+1}(z) = \kappa_{n+1} z\phi_n(z) + \phi_{n+1}(0)\,\phi_n^*(z). \tag{2.3}$$

Here the '*' is defined as

$$\phi_n^*(z) = \overline{a_n} + \overline{a_{n-1}}z + \cdots + \overline{a_0}\,z^n \tag{2.4}$$

if $\phi_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Notice that for $\phi_n(z)$, the coefficients are real as mentioned above.

Since we are interested in the leading coefficient κ_n, we set $\phi_n(z) = \kappa_n\,p_n(z)$, and for simplicity we denote $s = \sqrt{\lambda}$. Then (2.1) becomes

$$\int_{-\pi}^{\pi} p_n(\mathrm{e}^{\mathrm{i}\theta})\,\overline{p_m(\mathrm{e}^{\mathrm{i}\theta})}\,f(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi} = \delta_{nm}\frac{1}{\kappa_n^2} \qquad m, n \geqslant 0. \tag{2.5}$$

If we eliminate $\phi_n^*(z)$ in (2.2) and (2.3), we obtain

$$z\left(\frac{\kappa_{n+1}}{\kappa_n}\,\phi_n(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)}\,\frac{\kappa_{n-1}}{\kappa_n}\,\phi_{n-1}(z)\right) = \phi_{n+1}(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)}\,\phi_n(z). \tag{2.6}$$

By $\phi_n(z) = \kappa_n\,p_n(z)$, equation (2.6) becomes

$$z\left(p_n(z) - \frac{p_{n+1}(0)}{p_n(0)}\frac{\kappa_{n-1}^2}{\kappa_n^2}p_{n-1}(z)\right) = p_{n+1}(z) - \frac{p_{n+1}(0)}{p_n(0)}p_n(z). \tag{2.7}$$

For usage in later discussions, let us record this as a lemma, which is due to Szegö [1].

Lemma 1. *The orthogonal polynomials $p_n(z)$ defined by (2.5) on the unit circle in the complex plane satisfy the following recursion formula $n = 1, 2, \ldots$:*

$$z\big(p_n(z) + C_n\,p_{n-1}(z)\big) = p_{n+1}(z) + B_n\,p_n(z) \tag{2.8}$$

where

$$C_n = -\frac{p_{n+1}(0)}{p_n(0)}\frac{\kappa_{n-1}^2}{\kappa_n^2} \tag{2.9}$$

$$B_n = -\frac{p_{n+1}(0)}{p_n(0)}. \tag{2.10}$$

As a remark, the recursion formula for the orthogonal polynomials on the real line takes the form [1, 11]

$$zp_n = a_n\,p_{n+1} + b_n\,p_n + c_n\,p_{n-1}. \tag{2.11}$$

However, we have seen that on the unit circle with the weight $\mathrm{d}\mu = f(z)\,\mathrm{d}z/z$, the coefficient of p_{n-1} in the recursion formula contains z.

3. Spatially discrete equation for $\kappa_n(s)$

In this section, we use the orthogonal relation (2.5) and the recursion relation (2.8) to show that $\kappa_n(s)$ satisfies a spatially discrete equation (difference-differential equation). In the next section we show that the continuous limit of this spatially discrete equation gives a third-order ordinary differential equation, which is equivalent to the Painlevé II equation.

Let us write (2.5) in the form

$$\langle p_n, p_m\rangle \equiv \oint p_n(z)\,\overline{p_m(z)}\,f(z)\,\frac{\mathrm{d}z}{z} = \delta_{nm}\,h_n \qquad m, n \geqslant 0 \tag{3.1}$$

where $h_n = 2\pi\mathrm{i}/\kappa_n^2$, and the integral $\oint$ is taken over the unit circle oriented anticlockwise in the complex plane. We will use the notation $\mathrm{d}\mu = f(z)\,\mathrm{d}z/z$ for simplicity. First it is easy to show the following identities:

$$\oint z^k\,\frac{\partial\overline{p_m(z)}}{\partial\bar z}\,p_n(z)\,\mathrm{d}\mu = \oint z^{-k}\,\frac{\partial p_m(z)}{\partial z}\,\overline{p_n(z)}\,\mathrm{d}\mu \tag{3.2}$$

$$\oint z^k\,\overline{p_m(z)}\,p_n(z)\,\mathrm{d}\mu = \oint z^{-k}\,p_m(z)\,\overline{p_n(z)}\,\mathrm{d}\mu \tag{3.3}$$

by using $z = \mathrm{e}^{\mathrm{i}\theta}$, and θ replaced by $-\theta$. Recall $\overline{p_n(z)} = p_n(\bar z)$.

Now, consider (3.1) with $m = n$,

$$\begin{aligned} h_n &= \oint p_n(\mathrm{e}^{\mathrm{i}\theta})\,\overline{p_n(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{e}^{s(z+1/z)}\,\frac{\mathrm{d}z}{z} \\ &= \frac{1}{s}\oint p_n(\mathrm{e}^{\mathrm{i}\theta})\,\overline{p_n(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{e}^{s(1/z)}\,\frac{1}{z}\,\mathrm{d}(\mathrm{e}^{sz}). \end{aligned}$$

Then by integration by parts and using (3.2) and (3.3), we have

$$h_n = \frac{1}{s}\oint z^2\,\frac{\partial p_n(z)}{\partial z}\,\overline{p_n(z)}\,\mathrm{d}\mu + \oint z^2 p_n(z)\overline{p_n(z)}\,\mathrm{d}\mu + \frac{1}{s}\oint z p_n(z)\overline{p_n(z)}\,\mathrm{d}\mu. \tag{3.4}$$

As a remark for the calculation of this formula, the term where $p_n(z)$ is differentiated leads to an extra term like the first one in (3.4) with z^2 replaced by -1. It can be omitted due to the orthogonality.

Since $\overline{p_n(z)} = p(\bar z) = p(1/z)$ on the unit circle, by the Cauchy integral formula, the right-hand side of the above equation can be calculated, and the result involves all the coefficients of $p_n(z)$. This does not particularly help the investigation of κ_n. Here, by using the recursion formula and the orthonormal property, the right side of (3.4) can be expressed in terms of h_{n-1}, h_n, h_{n+1} and their first derivatives.

We need to consider two types of integrals,

$$\oint z^k\,\frac{\partial}{\partial z}p_n(z)\,\overline{p_n(z)}\,\mathrm{d}\mu \qquad\qquad \oint z^k\,p_n(z)\,\overline{p_n(z)}\,\mathrm{d}\mu$$

for (3.4), where k is integer. In this paper, we do not discuss the general formula. We just consider the integrals in (3.4), and the formulae are given in the following lemmas.

Lemma 2.

$$\oint z\,p_n(z)\,\overline{p_n(z)}\,\mathrm{d}\mu = (B_n - C_n)\,h_n \tag{3.5}$$

$n = 1, 2, \ldots.$

Proof. Multiply $\overline{p_n(z)}$ on both sides of (2.8). Then integrating on both sides yields the result. □

Lemma 3.

$$\oint z^2 p_n(z)\overline{p_n(z)}\,\mathrm{d}\mu = -(C_{n+1}(B_n - C_n) - (B_n - C_n)^2 + C_n(B_{n-1} - C_{n-1}))\,h_n \tag{3.6}$$

$n = 1, 2, \ldots.$

Proof. Let $I = \oint z^2\, p_n(z)\, \overline{p_n(z)}\,\mathrm{d}\mu$. By lemma 1, we have

$$z^2 p_n = p_{n+2} + B_{n+1} p_{n+1} - C_{n+1} z p_n + B_n z p_n - C_n z^2 p_{n-1}$$

which implies by (3.1)

$$I = \oint \big((-C_{n+1} + B_n)\, z p_n - C_n z^2 p_{n-1}\big)\,\overline{p_n(z)}\,\mathrm{d}\mu. \tag{3.7}$$

Again by lemma 1,

$$z^2 p_{n-1} = z p_n + B_{n-1} p_n + B_{n-1}^2 p_{n-1} - z B_{n-1} C_{n-1} p_{n-2} - C_{n-1} z^2 p_{n-2}.$$

Since $\langle p_{n-1}, p_n\rangle = \langle z\, p_{n-2}, p_n\rangle = 0$, we then obtain from (3.7)

$$\begin{aligned} I &= \oint \big\{(-C_{n+1} + B_n) z p_n - C_n (z p_n + B_{n-1} p_n - C_{n-1} z^2 p_{n-2})\big\}\overline{p_n}\,\mathrm{d}\mu \\ &= \oint \big\{(-C_{n+1} + B_n - C_n)(p_{n+1} + B_n p_n - z C_n p_{n-1}) \\ &\quad - C_n B_{n-1} p_n + C_n C_{n-1} z^2 p_{n-2}\big\}\overline{p_n}\,\mathrm{d}\mu \\ &= (-C_{n+1} + B_n - C_n)(B_n - C_n) h_n - C_n B_{n-1} h_n + C_n C_{n-1} h_n \end{aligned}$$

where $\langle z p_{n-1}, p_n\rangle = \langle z^2 p_{n-2}, p_n\rangle = h_n$. □

In order to evaluate the integral $\langle z^2\, \partial p_n/\partial z, p_n\rangle$ in (3.4), we need to consider the second leading coefficient of $p_n(z)$. Suppose

$$p_n(z) = z^n + b_{n-1} z^{n-1} + \cdots. \tag{3.8}$$

Because $z^2 \partial p_n(z)/\partial z = n z^{n+1} + (n-1) b_{n-1} z^n + \cdots$, we set

$$z^2 \frac{\partial p_n(z)}{\partial z} = n(p_{n+1} + \mu_n p_n + \cdots).$$

Then by

$$z^{n+1} + b_n z^n + \cdots + \mu_n (z^n + b_{n-1} z^{n-1} + \cdots) = z^{n+1} + \left(1 - \frac{1}{n}\right) b_{n-1} z^n + \cdots$$

it follows that

$$\mu_n = \left(1 - \frac{1}{n}\right) b_{n-1} - b_n. \tag{3.9}$$

Since $\langle z^2 \partial p_n/\partial z, p_n\rangle = n\mu_n h_n$, we have proved the following lemma.

Lemma 4.

$$\oint z^2 \frac{\partial p_n(z)}{\partial z}\overline{p_n(z)}\,\mathrm{d}\mu = (n(b_{n-1} - b_n) - b_{n-1}) h_n. \tag{3.10}$$

Because we want an equation for the leading coefficient κ_n, we need to express the second leading coefficient b_n in terms of κ_n. First, the b_n can be expressed in terms of B_n and C_n.

Lemma 5. *For the second leading coefficient b_{n-1} $(n = 1, 2, \ldots)$ of p_n defined by (3.8), there are the following properties:*

(a) $$b_{n-1} - b_n = B_n - C_n \tag{3.11}$$

(b) $$\frac{1}{s} b_{n-1} = -C_n(B_{n-1} - C_{n-1}) - \frac{h_n}{h_{n-1}}. \tag{3.12}$$

Proof. By lemma 1, $zp_n(z) = p_{n+1}(z) + B_n p_n(z) - C_n z p_{n-1}(z)$. By (3.8), $zp_n = p_{n+1} + (b_{n-1} - b_n)p_n + \cdots$. Then we have $(b_{n-1} - b_n)h_n = B_n h_n - C_n h_n$ by considering $\langle zp_n, p_n\rangle$ in the two ways.

To prove the second formula (3.12), consider

$$J = \oint p_{n+1}(z)\overline{p_n(z)}\, \mathrm{e}^{s(z+(1/z))}\mathrm{d}z. \tag{3.13}$$

Using integration by parts,

$$\begin{aligned} J &= \frac{1}{s} \oint p_{n+1}(z)\overline{p_n(z)}\, \mathrm{e}^{s/z}\, \mathrm{d}\mathrm{e}^{sz} \\ &= \frac{-1}{s} \oint \left(z \frac{\partial p_{n+1}}{\partial z} \right) \overline{p_n}\, \mathrm{d}\mu + \frac{1}{s} \oint p_{n+1} \left(\bar{z} \frac{\partial \overline{p_n}}{\partial \bar{z}} \right) \mathrm{d}\mu + \oint p_{n+1}(\bar{z}\overline{p_n})\, \mathrm{d}\mu \end{aligned}$$

where $\langle p_{n+1}, z\partial p_n/\partial z\rangle = 0$, and $\langle p_{n+1}, zp_n\rangle = h_{n+1}$. And $z\partial p_{n+1}/\partial z = (n+1)(z^{n+1} + n/(n+1)b_n z^n + \cdots)$. Set $z\partial p_{n+1}/\partial z = (n+1)(p_{n+1} + \gamma_n p_n + \cdots)$, which implies $\langle z\partial p_{n+1}/\partial z, p_n\rangle = (n+1)\gamma_n h_n$. Then by

$$p_{n+1} + \gamma_n p_n + \cdots = z^{n+1} + \frac{n}{n+1} b_n z^n + \cdots$$

we obtain $\gamma_n + b_n = [n/(n+1)]b_n$, so $\gamma_n = -(n+1)^{-1} b_n$. Thus

$$J = \frac{1}{s} b_n h_n + h_{n+1}. \tag{3.14}$$

On the other hand, the integral J can be evaluated by using the recursion formula (2.8)

$$zp_{n+1} = p_{n+2} + B_{n+1}p_{n+1} - C_{n+1}p_{n+1} - C_{n+1}B_n p_n + C_{n+1}C_n z p_{n-1}$$

which gives

$$J = \langle zp_{n+1}, p_n\rangle = -C_{n+1}(B_n - C_n)h_n. \tag{3.15}$$

By (3.14) and (3.15), we obtain (3.12). □

We have seen by the last four lemmas that the right-hand side of (3.4) can be expressed in terms of B_n and C_n. The following lemma gives the formulae of B_n, C_n in terms of κ_n.

Lemma 6. *The coefficients B_n, C_n in the recursion formula (see (2.9) and (2.10)) can be expressed as follows:*

(a) $$B_n - C_n = -\frac{(\kappa_n^2)_s}{2\kappa_n^2} \tag{3.16}$$

(b) $$B_n = -\frac{(\kappa_n^2)_s}{2(\kappa_n^2 - \kappa_{n-1}^2)} \tag{3.17}$$

(c) $$C_n = -\frac{\kappa_{n-1}^2}{2\kappa_n^2} \frac{(\kappa_n^2)_s}{\kappa_n^2 - \kappa_{n-1}^2} \tag{3.18}$$

where κ_n is the leading coefficient of ϕ_n defined by (1.1), and $(\kappa_n^2)_s = \mathrm{d}\kappa_n^2(s)/\mathrm{d}s$.

Proof. Since $h_n = \langle p_n, p_n\rangle$, we have

$$\frac{\mathrm{d}h_n(s)}{\mathrm{d}s} = \oint p_n(z)\overline{p_n(z)}\left(z+\frac{1}{z}\right)\mathrm{d}\mu = 2\oint z p_n\overline{p_n}\mathrm{d}\mu \tag{3.19}$$

where we have used $\langle \partial p_n/\partial s, p_n\rangle = \langle p_n, \partial p_n/\partial s\rangle = 0$, since $\deg(\partial p_n/\partial s) \leqslant n-1$, $\deg(\partial\overline{p_n}/\partial s) \leqslant n-1$. Then by lemma 1 and the notation $h_n = 2\pi \mathrm{i}/\kappa_n^2$, we have the first formula (3.16). By (2.9) and (2.10) $C_n = B_n\kappa_{n-1}^2/\kappa_n^2$. We then obtain (3.17) and (3.18). □

By lemmas 2–4, equation (3.4) is changed to

$$1 = \frac{n}{s}(b_{n-1}-b_n) - \frac{1}{s}b_{n-1} - C_{n+1}(B_n - C_n) + (B_n - C_n)^2 - C_n(B_{n-1}-C_{n-1}) + \frac{1}{s}(B_n - C_n). \tag{3.20}$$

By lemma 5, this equation further becomes

$$\frac{n+1}{s}(B_n - C_n) + \frac{h_n}{h_{n-1}} - 1 - C_{n+1}(B_n - C_n) + (B_n - C_n)^2 = 0. \tag{3.21}$$

Therefore, by lemma 6, we have proved the following theorem.

Theorem 1. *The leading coefficient $\kappa_n(s)$ of the orthonormal polynomials $\phi_n(z; s)$ defined by (1.1) satisfies the following spatially discrete equation (difference-differential equation):*

$$\frac{n+1}{2s}\frac{(\kappa_n^2)_s}{\kappa_n^2} - \frac{\kappa_{n-1}^2-\kappa_n^2}{\kappa_n^2} + \frac{1}{4}\frac{(\kappa_{n+1}^2)_s}{\kappa_{n+1}^2}\frac{(\kappa_n^2)_s}{\kappa_{n+1}^2-\kappa_n^2} - \frac{1}{4}\left(\frac{(\kappa_n^2)_s}{\kappa_n^2}\right)^2 = 0 \tag{3.22}$$

where $s=\sqrt{\lambda}$, and $(\kappa_n^2)_s = \mathrm{d}/\mathrm{d}s(\kappa_n^2)$.

This seems to be a new result for $\kappa_n(s)$, the leading coefficients of the orthonormal polynomials $\phi_n(z; s)$, with the weight $\exp\left(s(z+1/z)\right)\mathrm{d}z/z$ on the unit circle. In the next section, we show that as $n, s \to \infty$ and $2s/(n+1) \to 1$, this equation is reduced to a third-order ordinary differential equation which is equivalent to the Painlevé II equation.

4. Painlevé II equation

We have shown that κ_n^2 satisfies equation (3.22). As mentioned in section 1, κ_n^2 satisfies the boundary condition $\kappa_n^2(s) = 1 + \mathrm{o}(1)$, as $n \to \infty$ for all s. In this section, we compute the asymptotics of $\kappa_n^2 - 1$, as $n, s \to \infty$. We will see that the asymptotics involves the second Painlevé function.

Equation (3.22) has two independent variables n and s. To study the asymptotics, we use 'similarity' reduction, or the so-called double-scaling method. By comparing the first two terms in (3.22), we consider the case when $n+1$ and s are of the same order as they are large, and let

$$\frac{c_3 s}{n+1} = 1 + \frac{c_2 t(n,s)}{(n+1)^{\beta}} \tag{4.1}$$

$$\kappa_n^2 = 1 + \frac{c_1}{(n+1)^{\alpha}}R(T(n,s)) \tag{4.2}$$

where we assume $T(n,s) = t(n,s) + \epsilon(n,s)$, the leading term $t(n,s)$ is defined by (4.1), $\epsilon(n,s)$ is a smaller term as $n, s \to \infty$, and α, β will be discussed later. We want to determine the constants α, β, c_1, c_2, c_3, such that as $n, s \to \infty$, the difference-differential equation (3.22) is reduced to an ordinary differential equation of $R(t)$.

Let us consider the approximate expressions of $(\kappa_n^2)_s$, $\kappa_{n+1}^2 - \kappa_n^2$ and $(\kappa_{n+1}^2)_s - (\kappa_n^2)_s$ in terms of R' and the higher-order derivatives. Here R' means the limit of $(R(T + \Delta T) - R(T))/\Delta T$. The primary part of ΔT is Δt, and it can be calculated that as $n, s \to \infty$, and $c_3 s/(n+1) \to 1$, we have

$$t(n+1, s) - t(n, s) = -\frac{\beta}{c_2(n+1)^{1-\beta}} + \frac{c_3 s}{n+1}\frac{-1+\beta}{c_2(n+1)^{1-\beta}} + \mathrm{O}\left(\frac{1}{(n+1)^{2-\beta}}\right)$$
$$= -\frac{1}{c_2(n+1)^{1-\beta}} + \mathrm{O}\left(\frac{1}{n+1}\right).$$

We see that β should be chosen such that $0 < \beta < 1$, and then the rest term is $\mathrm{O}(1/(n+1))$. For the asymptotics of $\kappa_{n+1}^2 - \kappa_n^2$ and $(\kappa_{n+1}^2)_s - (\kappa_n^2)_s$, the higher-order terms must be considered in order to have all the terms in the first three leading orders in the expansion of left-hand side of (3.22). Therefore, we have the following:

$$t(n+1, s) - t(n, s) = \frac{c_4}{(n+1)^{1-\beta}} + \cdots$$
$$\kappa_{n+1}^2 - \kappa_n^2 = R'\frac{c_5}{(n+1)^{1+\alpha-\beta}} + R''\frac{c_6}{(n+1)^{2+\alpha-2\beta}} + R'''\frac{c_7}{(n+1)^{3+\alpha-3\beta}} + R\frac{c_8}{(n+1)^{1+\alpha}} + \cdots$$
$$\kappa_{n-1}^2 - \kappa_n^2 = -R'\frac{c_5}{(n+1)^{1+\alpha-\beta}} + R''\frac{c_6}{(n+1)^{2+\alpha-2\beta}}$$
$$-R'''\frac{c_7}{(n+1)^{3+\alpha-3\beta}} - R\frac{c_8}{(n+1)^{1+\alpha}} + \cdots$$
$$(\kappa_n^2)_s = \frac{c_1 c_3}{c_2}R'\frac{1}{(n+1)^{1+\alpha-\beta}} + \cdots$$
$$(\kappa_{n+1}^2)_s - (\kappa_n^2)_s = \frac{c_1 c_3}{c_2(n+1)^{1+\alpha-\beta}}\left(R''\frac{c_4}{(n+1)^{1-\beta}} + \frac{R'''}{2}\frac{c_4^2}{(n+1)^{2-2\beta}}\right) + \cdots$$

where

$$c_4 = -\frac{1}{c_2} \qquad c_5 = c_1 c_4 \qquad c_6 = \frac{1}{2}c_1 c_4^2 \tag{4.3}$$
$$c_7 = \tfrac{1}{6}c_1 c_4^3 \qquad c_8 = -c_1\alpha. \tag{4.4}$$

Now write (3.22) in the following form:

$$\frac{n+1}{2s}(\kappa_n^2)_s(\kappa_{n+1}^2 - \kappa_n^2) - (\kappa_{n-1}^2 - \kappa_n^2)(\kappa_{n+1}^2 - \kappa_n^2) + \frac{1}{4}(\kappa_n^2)_s(\kappa_{n+1}^2)_s$$
$$-\frac{1}{4}\frac{\kappa_{n+1}^2 - \kappa_n^2}{\kappa_{n+1}^2}(\kappa_n^2)_s(\kappa_{n+1}^2)_s - \frac{1}{4}\frac{((\kappa_n^2)_s)^2}{\kappa_n^2}(\kappa_{n+1}^2 - \kappa_n^2) = 0.$$

By substituting the asymptotic formulae above into this equation, we obtain

$$S_1 + S_2 + S_3 + S_4 + \mathrm{o}(1) = 0 \tag{4.5}$$

where

$$S_1 = \frac{c_1 c_3^2}{2c_2}\left(1 - \frac{c_2 t}{(n+1)^{\beta}}\right)\frac{R'}{(n+1)^{1+\alpha-\beta}}$$
$$\times\left\{\frac{c_5 R'}{(n+1)^{1+\alpha-\beta}} + \frac{c_6 R''}{(n+1)^{2+\alpha-2\beta}} + \frac{c_7 R'''}{(n+1)^{3+\alpha-3\beta}} + \frac{c_8 R}{(n+1)^{1+\alpha}}\right\}$$
$$S_2 = \frac{c_5^2(R')^2}{(n+1)^{2+2\alpha-2\beta}} + \frac{2c_5 c_7 R' R'''}{(n+1)^{4+2\alpha-4\beta}} + \frac{2c_5 c_8 R' R}{(n+1)^{2+2\alpha-\beta}} - \frac{c_6^2(R'')^2}{(n+1)^{4+2\alpha-4\beta}}$$

$$S_3 = \frac{c_1c_3}{4c_2}\frac{R'}{(n+1)^{1+\alpha-\beta}}\left\{\frac{c_1c_3}{c_2}\frac{R'}{(n+1)^{1+\alpha-\beta}} + \frac{c_1c_3}{c_2(n+1)^{1+\alpha-\beta}}\left(\frac{c_4R''}{(n+1)^{1-\beta}} + \frac{c_4^2R'''}{2(n+1)^{2-2\beta}}\right)\right\}$$

$$S_4 = -\frac{c_5}{4\kappa_{n+1}^2}\left(\frac{c_1c_3}{c_2}\right)^2\frac{(R')^3}{(n+1)^{3+3\alpha-3\beta}} - \frac{c_5}{4\kappa_n^2}\left(\frac{c_1c_3}{c_2}\right)^2\frac{(R')^3}{(n+1)^{3+3\alpha-3\beta}}.$$

We have seen that $0 < \beta < 1$. For α, since $\kappa_n^2(s) \to 1$, as $n \to \infty$ for all s, we have $\alpha > 0$. Consider the orders of the terms on the left-hand side of (4.5). The coefficients of $(R')^2$, $R'R''$, $R'R'''$, $(R'')^2$ have orders $2(1+\alpha-\beta)$, $2(1+\alpha-\beta)+(1-\beta)$, $2(1+\alpha-\beta)+2(1-\beta)$, $2(1+\alpha-\beta)+2(1-\beta)$, respectively. The coefficients of $t(R')^2$, RR', $(R')^3$ have orders $2(1+\alpha-\beta)+\beta$, $2(1+\alpha-\beta)+\beta$, $2(1+\alpha-\beta)+(1+\alpha-\beta)$, respectively. And the o(1) in (4.5) contains higher-order terms which do not concern us. To determine the values of α and β, the only choice is to set the coefficients of $R'R'''$, $(R'')^2$, $t(R')^2$, RR' and $(R')^3$ to be of the same order. So we have $2(1-\beta) = \beta = 1+\alpha-\beta$. The solution is unique $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$. So (4.5) becomes

$$\begin{aligned} A_1\frac{(R')^2}{(n+1)^{4/3}} &+ A_2\frac{R'R''}{(n+1)^{5/3}} \\ &+\left(A_3R'R + A_4R'R''' + A_5(R'')^2 + A_6t(R')^2 + A_7(R')^3\right)\frac{1}{(n+1)^2} \\ &+\mathrm{O}\left(\frac{1}{(n+1)^{7/3}}\right) = 0 \end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
A_1 &= \frac{c_3}{2}\frac{c_1c_3}{c_2}c_5 + c_5^2 + \frac{1}{4}\left(\frac{c_1c_3}{c_2}\right)^2 \\
A_2 &= \frac{c_3}{2}\frac{c_1c_3}{c_2}c_6 + \frac{1}{4}\left(\frac{c_1c_3}{c_2}\right)^2 c_4 \\
A_3 &= \frac{c_3}{2}\frac{c_1c_3}{c_2}c_8 + 2c_5c_8 \\
A_4 &= \frac{c_3}{2}\frac{c_1c_3}{c_2}c_7 + 2c_5c_7 + \frac{1}{8}\left(\frac{c_1c_3}{c_2}\right)^2 c_4^2 \\
A_5 &= -c_6^2 \\
A_6 &= -\tfrac{1}{2}c_1c_3^2c_5 \\
A_7 &= -\frac{1}{2}\left(\frac{c_1c_3}{c_2}\right)^2 c_5.
\end{aligned}$$

The A_js ($j = 1, 2, \ldots, 7$) can be expressed in terms of c_1, c_2, c_3 by using (4.3) and (4.4), and it is seen that $A_2 = 0$.

Look at equation (4.6). As $n \to \infty$, the leading term gives an equation $(1 - c_3^2/4)(R')^2 = 0$. If we choose $c_3^2 \neq 4$, then $R' = 0$, which implies $\kappa_n^2(s) = 1 + \text{constant}/(n+1)^{1/3} + \cdots$, as $n, s \to \infty$, $c_3\, s/(n+1) \to 1$. If we choose $c_3^2 = 4$, then $A_1 = A_3 = 0$, and as $n \to \infty$, equation (4.6) is reduced to

$$A_4\, R'R''' + A_5\,(R'')^2 + A_6\, t(R')^2 + A_7\,(R')^3 = 0. \tag{4.7}$$

In [2], Tracy and Widom discussed two forms of the Painlevé II equation

$$\frac{1}{2}\frac{R'''}{R'} - \frac{1}{2}\frac{(R'')^2}{(R')^2} - \frac{R}{R'} + R' = 0 \tag{4.8}$$

$$(R'')^2 + 4R'((R')^2 - tR' + R) = 0 \tag{4.9}$$

where $R'(t) = -q(t)^2$, and $q(t)$ satisfies the original Painlevé II $q'' = tq + 2q^3$. Equation (4.9) is called the Jimbo–Miwa–Okamoto σ-form for Painlevé II. Eliminating R in (4.8) and (4.9) gives another form,

$$-2R'R''' + (R'')^2 + 4t(R')^2 - 8(R')^3 = 0. \tag{4.10}$$

In (4.7), it can be calculated that $A_4 = -2A_5$ since $c_3^2 = 4$. Let $A_6 = 4A_5$ and $A_7 = -8A_5$. These two equations can be written as algebraic equations of c_1, c_2 by using (4.3) and (4.4), and the solution is

$$c_1 = -2^{1/3} \tag{4.11}$$

$$c_2 = -\frac{1}{2^{1/3}}. \tag{4.12}$$

For the c_3, since we consider positive n and s, c_3 is positive, i.e. $c_3 = 2$. That implies that if we choose $c_1 = -2^{1/3}, c_2 = -1/2^{1/3}, c_3 = 2$ in (4.1) and (4.2), the spatially discrete equation (3.22) in theorem 1 is reduced to the Painlevé II equation (4.10). That is why we call (3.22) a spatially discrete Painlevé II equation. And it is seen that $\epsilon(n, s) = \mathrm{O}(1/(n+1)^{1/3})$ because the last term in (4.6) is $\frac{1}{3}$ order higher than the preceding term.

Therefore, we have a formal proof of the following theorem which was first proved by Baik *et al* [3] by studying the corresponding Riemann–Hilbert problem.

Theorem 2. *As $n, \sqrt{\lambda} \to \infty$, and $2\sqrt{\lambda}/(n+1) \to 1$, $\kappa_n^2(\lambda)$ has the following asymptotic formula:*

$$\kappa_n^2(\lambda) = 1 - \frac{2^{1/3}}{(n+1)^{1/3}} R(t) + \mathrm{O}\left(\frac{1}{(n+1)^{2/3}}\right) \tag{4.13}$$

where t is defined by $2\sqrt{\lambda}/(n+1) = 1 - t/[2^{1/3}(n+1)^{2/3}]$, $R'(t) = -q^2(t)$, and $q(t)$ satisfies Painlevé II $q'' = tq + 2q^3$.

As discussed in [3], the Painlevé II function $q(t)$ in theorem 2 satisfies the boundary condition $q(t) \sim -\mathrm{Ai}(t)$, as $t \to \infty$, where $\mathrm{Ai}(t)$ is the Airy function. This boundary condition can also be obtained by the asymptotics of κ_n in terms of an exponential function [1] and the Painlevé II equation that $q(t)$ satisfies. The Painlevé II function $q(t)$ with this boundary condition is discussed by Hastings and McLeod in [12].

Finally, by lemmas 5 and 6 and theorem 2, we obtain the asymptotics for the second leading coefficient of $\phi_n(z)$.

Theorem 3. *For the polynomial $\phi_n(z;\lambda) = \kappa_n(\lambda)\,(z^n + b_{n-1}(\lambda)\,z^{n-1} + \cdots)$ defined by (1.1), the second leading coefficient $\kappa_n(\lambda)\,b_{n-1}(\lambda)$ has the asymptotic formula*

$$\frac{\kappa_n(\lambda)\,b_{n-1}(\lambda)}{\sqrt{\lambda}} = -1 + \frac{1}{2^{2/3}(n+1)^{1/3}} R(t) + \mathrm{O}\left(\frac{1}{(n+1)^{2/3}}\right) \tag{4.14}$$

as $n, \sqrt{\lambda} \to \infty$, and $2\sqrt{\lambda}/(n+1) \to 1$, where t and $R(t)$ are the same as in theorem 2.

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